Joschka Roffe
Stuart Ferguson

Tutorial 9

Problem 1: Pauli Commutation Relations

1a. Consider the two Pauli operators $P \in \mathcal{P}^{\otimes n}$ and $G \in \mathcal{P}^{\otimes n}$. These operators are said to intersect trivially at position i if $P_i = G_i$ or $P_i, G_i = I$. They intersect non-trivially if $P_i \neq G_i$ and $P_i, G_i \neq I$. Show that P and G will commute if they intersect non-trivially in an even number of locations and anti-commute if they intersect in an odd number of locations.

1b. Do the Pauli operators $X_1Z_2Y_5$ and $X_2Y_5X_7$ commute or anti-commute?

1c. Do the Pauli operators X_1Z_2 and Z_1X_2 commute or anti-commute?

Problem 2: The two-qubit repetition code for phase flips

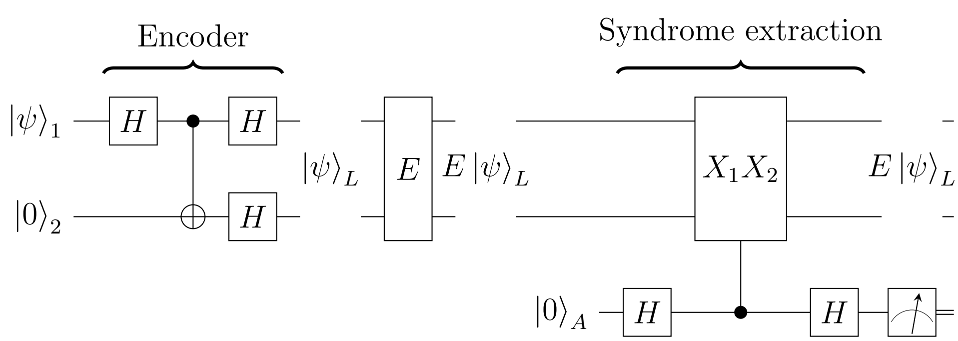

Figure 1: The two-qubit repetition code for phase flips

Figure 1 shows the two-qubit repetition code protocol for detecting phase-flip errors.

2a. What are the $|0\rangle_L$ and $|1\rangle_L$ logical basis states of this code?

2b. Show that the stabiliser generator X_1X_2 acts as the identity on the basis states.

2c. Show that immediately before the measurement of auxiliary qubit A the system is in the following state:

$$\frac{1}{2}(I + X_1X_2)E|\psi\rangle_L|0\rangle_A + \frac{1}{2}(I - X_1X_2)E|\psi\rangle_L|1\rangle_A$$

2d. Show that the measurement of auxiliary qubit A_1 yields '0' if $[E, X_1X_2] = 0$ and '1' if $\{E, X_1X_2\} = 0$.

2e. Complete syndrome table (Tab 1).

2f. Identify an X_L and Z_L logical operator for this code. Show that these operators have the correct action on the logical basis states.

Error	$\mathbf{s}_1$
$I_1 \otimes I_2$	
$X_1 \otimes I_2$	
$I_1 \otimes X_2$	
$X_1 \otimes I_2$	
$X_1 \otimes X_2$	
$I_1 \otimes Z_2$	
$Z_1 \otimes I_I$	
$Z_1 \otimes Z_2$	

Table 1: Syndrome table for the 2-qubit repetition code for phase flips.

2g. What is the distance of this code?

Problem 3: The Five-Qubit Code

The five-qubit code (also known as the 'perfect code') is defined by the stabiliser group $\mathcal{S}$ generated by $\langle S \rangle$:

$$\mathcal{S} = \langle S \rangle = \left\langle \begin{matrix} X_1 Z_2 Z_3 X_4 I_5 \\ I_1 X_2 Z_3 Z_4 X_5 \\ X_1 I_2 X_3 Z_4 Z_5 \\ Z_1 X_2 I_3 X_4 Z_5 \end{matrix} \right\rangle$$

3a. How many logical qubits are encoded by this code?

3b. The logical basis states of the five-qubit code are given below.

$$\begin{aligned} |0_{\mathrm{L}}\rangle = \frac{1}{4}(&|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\ &-|11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle), \end{aligned}$$

$$\begin{aligned} |1\rangle_L = \frac{1}{4}(&|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle \\ &-|00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle). \end{aligned}$$

Show that both $X_L = X_1 X_2 X_3 X_4 X_5$ and $Z_L = Z_1 Z_2 Z_3 Z_4 Z_5$ are a valid choice of logical operators for the code.

3c. Complete the single-qubit syndrome table for this code:

Error	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\mathbf{s}_4$
$X_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$Y_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$Z_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes X_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes Y_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes Z_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes X_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes Y_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes Z_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes X_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes Y_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes Z_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes X_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes Y_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes Z_5$				

Table 2: Single-Qubit Syndrome Table (Tab 2) for the Five-Qubit Code.

3d. Explain why this is a correction code with distance $d \geq 3$.

3e. Find a pair of X_L and Z_L logical operators of weight 3.

3f. What are the $[[n, k, d]]$ parameters of this code?

Problem 4: Concatenating the $[[3, 1, 1]]$ phase-flip code into the $[[4, 2, 2]]$ code

Consider the $[[4, 2, 2]]$ code defined by the stabiliser group $\mathcal{S}$ generated by $\langle S \rangle$:

$$\mathcal{S} = \langle S \rangle = \left\langle \begin{matrix} X_1X_2X_3X_4 \\ Z_1Z_2Z_3Z_4 \end{matrix} \right\rangle,$$

and logical operator basis:

$$\mathcal{L} = \left\langle \begin{matrix} X_{L_1} = X_1I_2X_3I_4 \\ X_{L_2} = X_1X_2I_3I_4 \\ Z_{L_1} = Z_1Z_2I_3I_4 \\ Z_{L_2} = Z_1I_2Z_3I_4 \end{matrix} \right\rangle.$$

Consider also the $[[3, 1, 1]]$ phase-flip code defined by the stabiliser group $\mathcal{S}$ generated by $\langle S \rangle$:

$$\mathcal{S} = \langle S \rangle = \left\langle \begin{matrix} X_1X_2I_3 \\ I_1X_2X_3 \end{matrix} \right\rangle,$$

and logical operator basis:

$$\mathcal{L} = \left\langle \begin{matrix} X_L = X_1 I_2 I_3 \\ Z_L = Z_1 Z_2 Z_3 \end{matrix} \right\rangle.$$

4a. Write a generating set of the stabilisers of code obtained by concatenating the $[[3,1,1]]$ phase-flip code into each of the qubits of the $[[4,2,2]]$ code. E.g. replace each qubit i of the $[[4,2,2]]$ code by three qubits i_1, i_2, i_3 that form a block encoded with the $[[3,1,1]]$ phase-flip code.

4b. How many logical qubits are encoded by this concatenated code?

4c. Find a basis of logical operators for this concatenated code.

4d. What is the distance of this concatenated code?

4e. What are the code parameters $[[n,k,d]]$ of this concatenated code?

4f. The d_X distance and d_Z distance of a code are defined as the minimum weight of a logical operator consisting exclusively of Pauli X operators and Pauli Z operators respectively. What are the X-distance and Z-distance of this concatenated code?

Problem 5: The 2×2 Surface Code

5a. Figure 2 shows the Tanner graph for a surface code defined over 5 qubits. List the four stabiliser generators that are measured by this code.

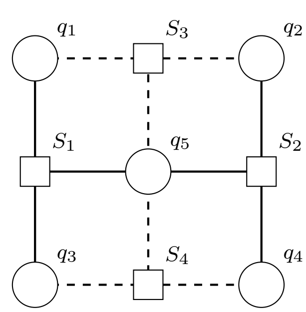

Figure 2: The five-qubit surface code. Dashed edges denote Z-type checks and solid edges X-type checks.

5b. How many logical qubits does this code encode?

5c. This code has distance $d = 2$. Find the logical operator pair Z_L, X_L.

5d. Explain why this code is a detection code and not a correction code.

5e. What are the $[[n, k, d]]$ parameters of this code?

5f. The d_i distance of a code is defined as the minimum weight of a logical operator consisting exclusively of Pauli i operators (where $i \in \{X, Y, Z\}$). Find d_X, d_Y, and d_Z for this code.

Problem 6: The 4×4 Surface Code

6a. Figure 3 shows the Tanner graph for a distance-4 surface code. Two X-errors have occurred on qubits q_{20} and q_6 activating a non-zero syndrome measurement for stabilisers S_{17} and S_{19}. Explain why $\mathcal{R} = X_6X_{20}$ and $\mathcal{R}' = X_{10}X_{21}$ are both suitable recovery operations.

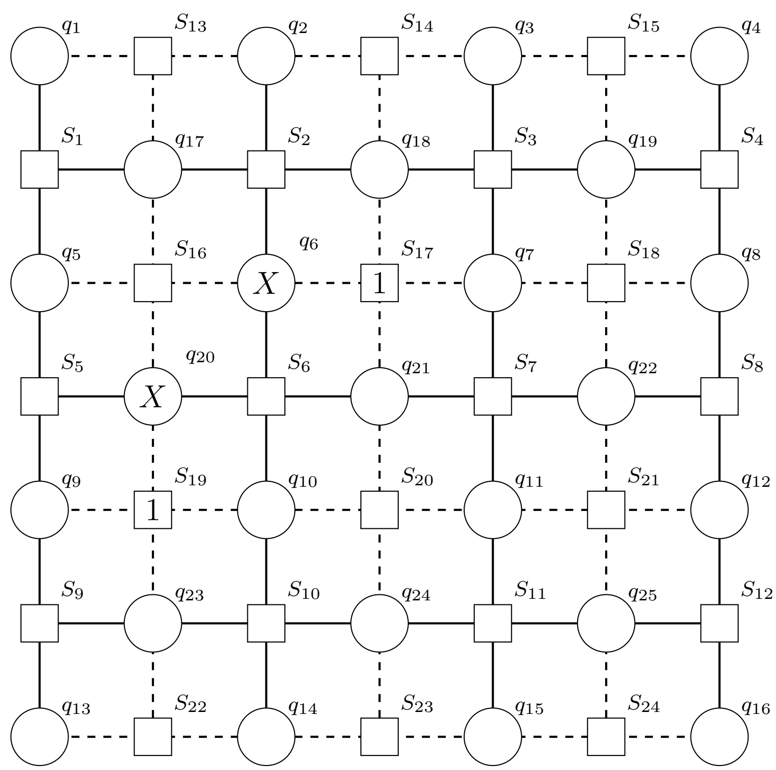

Figure 3: The distance-4 surface code. Dashed edges denote Z-type checks and solid edges X-type checks.

6b. The recovery operator $\mathcal{R}'' = X_7X_8X_9$ would also reset the total syndrome of the surface code. Explain why this is not a suitable recovery operator.

Problem 7: The Rotated Surface Code

The Tanner graph for a 5×5 rotated surface code is shown in Figure 4.

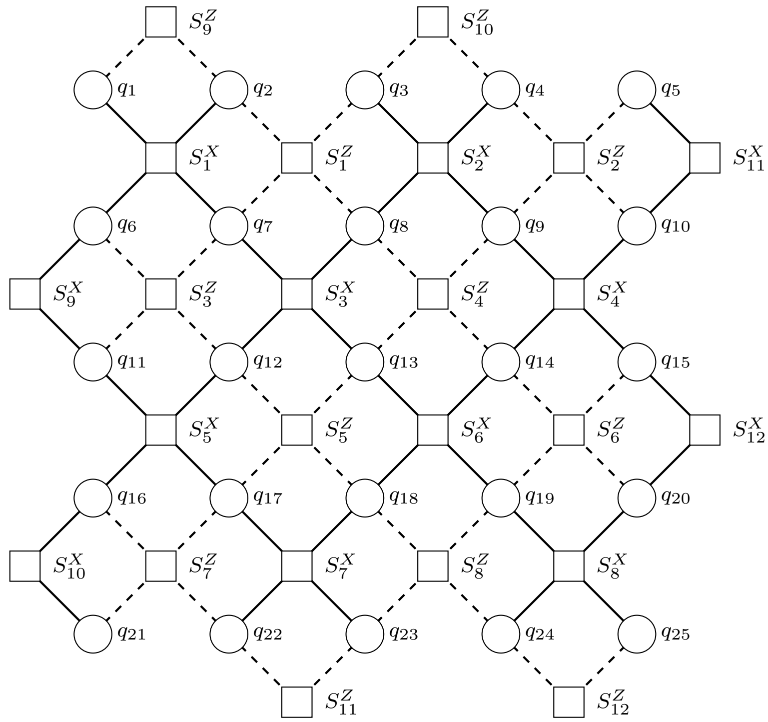

Figure 4: The Tanner graph for a 5×5 rotated surface code. Dashed lines represent Z-type Pauli operators and solid X-type Pauli operators.

7a. How many logical qubits does this code encode?

7b. Find weight-5 logical operators X_L and Z_L for this code.

7c. The *footprint* of a QEC code is defined as the total qubit count: data qubits plus auxiliary/measurement qubits. What is the ratio of footprints for the distance-d rotated surface code to the distance-d standard surface code?

Problem 8: Implementing the 7-qubit Steane Code in Pennylane

The 7-qubit Steane code is defined by the stabiliser group $\mathcal{S}$ generated by:

$$\mathcal{S} = \langle S \rangle = \left\langle \begin{matrix} I_1 I_2 I_3 X_4 X_5 X_6 X_7 \\ I_1 X_2 X_3 I_4 I_5 X_6 X_7 \\ X_1 I_2 X_3 I_4 X_5 I_6 X_7 \\ I_1 I_2 I_3 Z_4 Z_5 Z_6 Z_7 \\ I_1 Z_2 Z_3 I_4 I_5 Z_6 Z_7 \\ Z_1 I_2 Z_3 I_4 Z_5 I_6 Z_7 \end{matrix} \right\rangle,$$

8a. Show that the $|0\rangle_L$ logical state of the Steane code can be prepared using the following operation on the blank state:

$$|0\rangle_L = \prod_{P \in \langle S \rangle} \frac{I + P_i}{\sqrt{2^{|\langle S \rangle|}}} |0000000\rangle$$

8b. The circuit shown in Figure 5 can be used to map the blank state $|0000000\rangle$ onto the $+1$ eigenspace of the X-type stabiliser $IIIXXXX$.

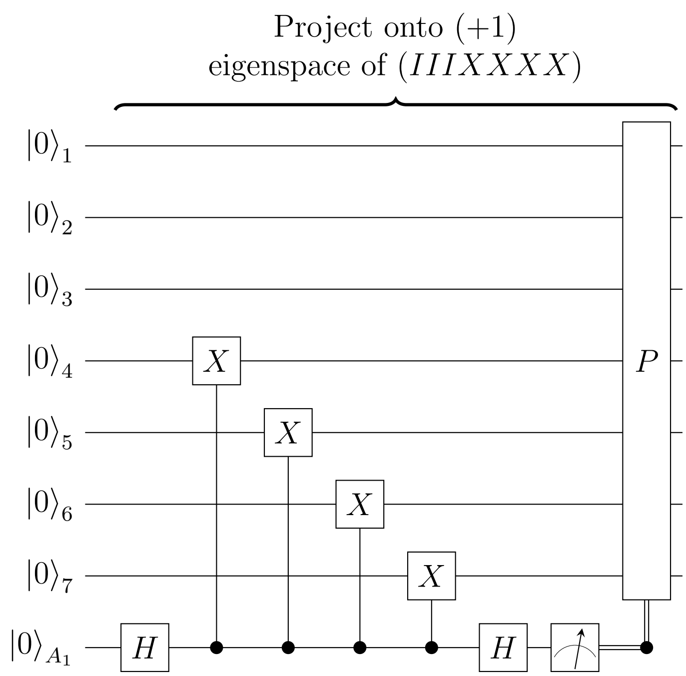

Figure 5: Circuit to map the blank state $|0000000\rangle$ onto the $+1$ eigenspace of the X-type stabiliser $IIIXXXX$.

Explain why the classical feedback is required to ensure the output state is in the $+1$ eigenspace. Find an appropiate form of the feedback operation P.

8c. Implement the full encoding circuit for the 7-qubit Steane code in Pennylane. You may use the circuit from Figure 5 as a subroutine and repeat for all stabilisers. *Hint:* You can use the function `qml.cond` to implement classical feedback based on measurement results. To verify that the stabiliser readout is deterministic, implement a full round of stabiliser measurements after the encoding circuit. Then check that all stabiliser measurements are deterministally by measuring out the auxiliary qubits with `qml.sample` for multiple shots.

8d. By inserting Pauli-errors after the encoding circuit, verify that your implementation can detect all single-qubit errors by creating the corresponding syndrome table.