Joschka Roffe
Stuart Ferguson

Tutorial 9

Problem 1: Pauli Commutation Relations

1a. Consider the two Pauli operators $P \in \mathcal{P}^{\otimes n}$ and $G \in \mathcal{P}^{\otimes n}$. These operators are said to intersect trivially at position i if $P_i = G_i$ or $P_i, G_i = I$. They intersect non-trivially if $P_i \neq G_i$ and $P_i, G_i \neq I$. Show that P and G will commute if they intersect non-trivially in an even number of locations and anti-commute if they intersect in an odd number of locations.

Solution 1a:

1. Let N be the number of qubits where P_i and G_i both act non-trivially ($P_i, G_i \neq I$) and $P_i \neq G_i$.
2. At each such qubit, P_i and G_i are different Pauli matrices, so they anti-commute: $P_i G_i = -G_i P_i$.
3. The total commutation factor is $(-1)^N$: $PG = (-1)^N GP$.
4. If N is even, $(-1)^N = 1$, so P and G commute.
5. If N is odd, $(-1)^N = -1$, so P and G anti-commute.

1b. Do the Pauli operators $X_1 Z_2 Y_5$ and $X_2 Y_5 X_7$ commute or anti-commute?

Solution 1b: They anti-commute as they intersect non-trivially only once on qubit 2.

1c. Do the Pauli operators $X_1 Z_2$ and $Z_1 X_2$ commute or anti-commute?

Solution 1c: They commute as they intersect non-trivially an even number of times on qubits 1 and 2.

Problem 2: The two-qubit repetition code for phase flips

Figure 1 shows the two-qubit repetition code protocol for detecting phase-flip errors.

2a. What are the $|0\rangle_L$ and $|1\rangle_L$ logical basis states of this code?

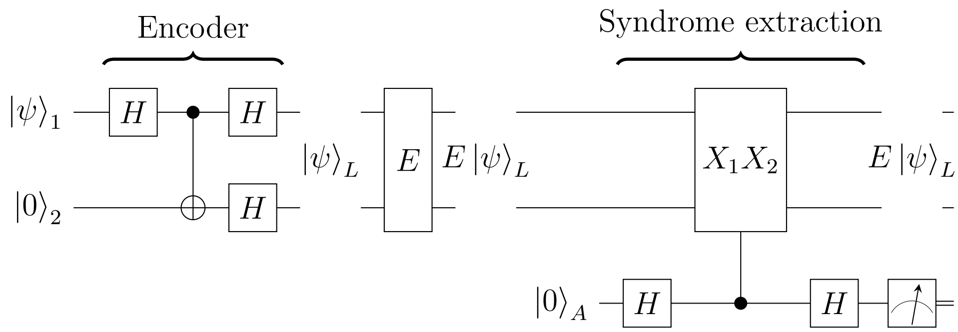

Figure 1: The two-qubit repetition code for phase flips

Solution 2a: We can determine the logical basis states by setting the encoder input $|\psi\rangle_1$ to $|0\rangle$ and $|1\rangle$ respectively. The output of the encoder then gives the logical basis states, $|0\rangle_L = \frac{1}{\sqrt{2}}(|{++}\rangle + |{--}\rangle)$ and $|1\rangle_L = \frac{1}{\sqrt{2}}(|{++}\rangle - |{--}\rangle)$.

2b. Show that the stabiliser generator X_1X_2 acts as the identity on the basis states.

Solution 2b: The Pauli-X operator acts as follows on the conjugate basis states: $X|+\rangle = |+\rangle$ and $X|-\rangle = (-1)|-\rangle$. We can use this to compute the action of X_1X_2 on the logical basis states:

$$\begin{aligned} X_1X_2|0\rangle_L &= X_1X_2\frac{1}{\sqrt{2}}(|{++}\rangle + |{--}\rangle) = \frac{1}{\sqrt{2}}(X_1X_2|{++}\rangle + X_1X_2|{--}\rangle) \\ &= \frac{1}{\sqrt{2}}(|{++}\rangle + (-1)(-1)|{--}\rangle) = \frac{1}{\sqrt{2}}(|{++}\rangle + |{--}\rangle) = |0\rangle_L \end{aligned}$$

$$\begin{aligned} X_1X_2|1\rangle_L &= X_1X_2\frac{1}{\sqrt{2}}(|{++}\rangle - |{--}\rangle) = \frac{1}{\sqrt{2}}(X_1X_2|{++}\rangle - X_1X_2|{--}\rangle) \\ &= \frac{1}{\sqrt{2}}(|{++}\rangle - (-1)(-1)|{--}\rangle) = \frac{1}{\sqrt{2}}(|{++}\rangle - |{--}\rangle) = |1\rangle_L \end{aligned}$$

Therefore, X_1X_2 acts as the identity on both logical basis states, confirming it is a valid stabiliser.

2c. Show that immediately before the measurement of auxiliary qubit A the system is in the following state:

$$\frac{1}{2}(I + X_1X_2)E\left|\psi\right\rangle_L\left|0\right\rangle_A + \frac{1}{2}(I - X_1X_2)E\left|\psi\right\rangle_L\left|1\right\rangle_A$$

Solution 2c:

We start with the state after the error has occurred:

$$E\left|\psi\right\rangle_L\left|0\right\rangle_A$$

where E is the error operator acting on the data qubits, $\left|\psi\right\rangle_L$ is the encoded logical state, and $\left|0\right\rangle_A$ is the initial state of the auxiliary qubit.

Step 1: Apply Hadamard Gate to the Auxiliary Qubit

The Hadamard gate transforms the auxiliary qubit as follows:

$$\left|0\right\rangle_A \xrightarrow{H} \frac{1}{\sqrt{2}}(\left|0\right\rangle_A + \left|1\right\rangle_A)$$

The combined state becomes:

$$E\left|\psi\right\rangle_L\left(\frac{1}{\sqrt{2}}(\left|0\right\rangle_A + \left|1\right\rangle_A)\right) = \frac{1}{\sqrt{2}}E\left|\psi\right\rangle_L\left|0\right\rangle_A + \frac{1}{\sqrt{2}}E\left|\psi\right\rangle_L\left|1\right\rangle_A$$

Step 2: Apply Controlled-X_1X_2 Gate

The controlled-X_1X_2 gate applies the X_1X_2 operator to the data qubits when the auxiliary qubit is in state $\left|1\right\rangle_A$:

$$\frac{1}{\sqrt{2}}E\left|\psi\right\rangle_L\left|0\right\rangle_A + \frac{1}{\sqrt{2}}X_1X_2E\left|\psi\right\rangle_L\left|1\right\rangle_A$$

Step 3: Apply Hadamard Gate to the Auxiliary Qubit Again

Applying the Hadamard gate to the auxiliary qubit transforms the states:

$$\left|0\right\rangle_A \xrightarrow{H} \frac{1}{\sqrt{2}}(\left|0\right\rangle_A + \left|1\right\rangle_A)$$
$$\left|1\right\rangle_A \xrightarrow{H} \frac{1}{\sqrt{2}}(\left|0\right\rangle_A - \left|1\right\rangle_A)$$

The total state becomes:

$$\begin{aligned}&\frac{1}{\sqrt{2}}E\left|\psi\right\rangle_L\left(\frac{1}{\sqrt{2}}(\left|0\right\rangle_A + \left|1\right\rangle_A)\right) + \frac{1}{\sqrt{2}}X_1X_2E\left|\psi\right\rangle_L\left(\frac{1}{\sqrt{2}}(\left|0\right\rangle_A - \left|1\right\rangle_A)\right)\\ &= \frac{1}{2}E\left|\psi\right\rangle_L(\left|0\right\rangle_A + \left|1\right\rangle_A) + \frac{1}{2}X_1X_2E\left|\psi\right\rangle_L(\left|0\right\rangle_A - \left|1\right\rangle_A)\end{aligned}$$

Step 4: Combine Like Terms

Grouping terms with $|0\rangle_A$ and $|1\rangle_A$:

$$\text{Coefficient of } |0\rangle_A : \frac{1}{2}E|\psi\rangle_L + \frac{1}{2}X_1X_2E|\psi\rangle_L = \frac{1}{2}(I + X_1X_2)E|\psi\rangle_L$$
$$\text{Coefficient of } |1\rangle_A : \frac{1}{2}E|\psi\rangle_L - \frac{1}{2}X_1X_2E|\psi\rangle_L = \frac{1}{2}(I - X_1X_2)E|\psi\rangle_L$$

Therefore, the state immediately before the measurement of the auxiliary qubit A is:

$$\frac{1}{2}(I + X_1X_2)E|\psi\rangle_L|0\rangle_A + \frac{1}{2}(I - X_1X_2)E|\psi\rangle_L|1\rangle_A$$

This is the desired expression, showing that the system is in the given state before measuring the auxiliary qubit.

2d. Show that the measurement of auxiliary qubit A_1 yields '0' if $[E, X_1X_2] = 0$ and '1' if $\{E, X_1X_2\} = 0$.

Solution 2d:

From part 2c, immediately before the measurement of the auxiliary qubit A, the state of the system is:

$$\frac{1}{2}(I + X_1X_2)E|\psi\rangle_L|0\rangle_A + \frac{1}{2}(I - X_1X_2)E|\psi\rangle_L|1\rangle_A .$$

The probability of measuring A in state $|0\rangle_A$ is proportional to the squared norm of the coefficient of $|0\rangle_A$:

$$P(0) = \left\| \frac{1}{2}(I + X_1X_2)E|\psi\rangle_L \right\|^2 .$$

Similarly, the probability of measuring A in state $|1\rangle_A$ is:

$$P(1) = \left\| \frac{1}{2}(I - X_1X_2)E|\psi\rangle_L \right\|^2 .$$

Case 1: If E **commutes** with X_1X_2, i.e., $[E, X_1X_2] = 0$, then:

$$X_1X_2E|\psi\rangle_L = EX_1X_2|\psi\rangle_L = E|\psi\rangle_L ,$$

since X_1X_2 stabilizes $|\psi\rangle_L$. Thus:

$$(I + X_1X_2)E|\psi\rangle_L = (I + I)E|\psi\rangle_L = 2E|\psi\rangle_L ,$$

and

$$(I - X_1X_2)E\left|\psi\right\rangle_L = (I - I)E\left|\psi\right\rangle_L = 0.$$

Therefore, $P(0) = \|E\left|\psi\right\rangle_L\|^2$ and $P(1) = 0$. The measurement yields outcome 0.

Case 2: If E **anticommutes** with X_1X_2, i.e., $\{E, X_1X_2\} = 0$, then:

$$X_1X_2E\left|\psi\right\rangle_L = -EX_1X_2\left|\psi\right\rangle_L = -E\left|\psi\right\rangle_L.$$

Thus:

$$(I + X_1X_2)E\left|\psi\right\rangle_L = (I - I)E\left|\psi\right\rangle_L = 0,$$

and

$$(I - X_1X_2)E\left|\psi\right\rangle_L = (I + I)E\left|\psi\right\rangle_L = 2E\left|\psi\right\rangle_L.$$

Therefore, $P(0) = 0$ and $P(1) = \|E\left|\psi\right\rangle_L\|^2$. The measurement yields outcome 1.

Thus, the measurement of auxiliary qubit A yields 0 if $[E, X_1X_2] = 0$ and 1 if $\{E, X_1X_2\} = 0$.

2e. Complete syndrome table (Tab 1).

Error	$\mathbf{s}_1$
$I_1 \otimes I_2$	
$X_1 \otimes I_2$	
$I_1 \otimes X_2$	
$X_1 \otimes I_2$	
$X_1 \otimes X_2$	
$I_1 \otimes Z_2$	
$Z_1 \otimes I_I$	
$Z_1 \otimes Z_2$	

Table 1: Syndrome table for the 2-qubit repetition code for phase flips.

Solution 2e:

We complete the syndrome table by determining whether each error operator E commutes or anticommutes with the stabiliser X_1X_2. The syndrome bit $\mathbf{s}_1$ is 0 if E commutes with X_1X_2 and 1 if it anticommutes. The completed syndrome table is shown in Tab 2.

Error	$\mathbf{s}_1$
$I_1 \otimes I_2$	0
$X_1 \otimes I_2$	0
$I_1 \otimes X_2$	0
$X_1 \otimes X_2$	0
$I_1 \otimes Z_2$	1
$Z_1 \otimes I_2$	1
$Z_1 \otimes Z_2$	0

Table 2: Solution: syndrome table

Explanation:

- For errors I_1I_2, X_1I_2, I_1X_2, and X_1X_2, they either act trivially or identically on qubits where the stabiliser acts, resulting in commutation ($\mathbf{s}_1 = 0$).
- Errors involving Z operators (I_1Z_2, Z_1I_2) anticommute with X on the same qubit. Since they differ on one qubit where both act non-trivially and differently, they anticommute ($\mathbf{s}_1 = 1$).
- For Z_1Z_2, the error anticommutes with X on both qubits (an even number), so the total effect is commutation ($\mathbf{s}_1 = 0$).

2f. Identify an X_L and Z_L logical operator for this code. Show that these operators have the correct action on the logical basis states.

Solution 2f:

Identifying Logical Operators:

- **Logical Z operator (Z_L):** We choose $Z_L = Z_1Z_2$.
- **Logical X operator (X_L):** We choose $X_L = X_1$ or $X_L = X_2$.

Verification of Correct Action on Logical Basis States:

Action of Z_L on Logical States:

Recall that $Z\,|+\rangle = |-\rangle$ and $Z\,|-\rangle = |+\rangle$. We have:

$$Z_L \left|0\right\rangle_L = Z_1 Z_2 \frac{1}{\sqrt{2}}(\left|{++}\right\rangle + \left|{--}\right\rangle) = \frac{1}{\sqrt{2}}(\left|{--}\right\rangle + \left|{++}\right\rangle) = \left|0\right\rangle_L ,$$
$$Z_L \left|1\right\rangle_L = Z_1 Z_2 \frac{1}{\sqrt{2}}(\left|{++}\right\rangle - \left|{--}\right\rangle) = \frac{1}{\sqrt{2}}(\left|{--}\right\rangle - \left|{++}\right\rangle) = -\frac{1}{\sqrt{2}}(\left|{++}\right\rangle - \left|{--}\right\rangle) = -\left|1\right\rangle_L .$$

Therefore, Z_L leaves $\left|0\right\rangle_L$ unchanged and introduces a phase -1 to $\left|1\right\rangle_L$, acting as a logical Z operator.

Action of X_L on Logical States:

Recall that $X\left|+\right\rangle = \left|+\right\rangle$ and $X\left|-\right\rangle = -\left|-\right\rangle$. We have:

$$X_L \left|0\right\rangle_L = X_1 \frac{1}{\sqrt{2}}(\left|{++}\right\rangle + \left|{--}\right\rangle) = \frac{1}{\sqrt{2}}(\left|+\right\rangle \otimes \left|+\right\rangle - \left|-\right\rangle \otimes \left|-\right\rangle) = \frac{1}{\sqrt{2}}(\left|{++}\right\rangle - \left|{--}\right\rangle) = \left|1\right\rangle_L ,$$
$$X_L \left|1\right\rangle_L = X_1 \frac{1}{\sqrt{2}}(\left|{++}\right\rangle - \left|{--}\right\rangle) = \frac{1}{\sqrt{2}}(\left|+\right\rangle \otimes \left|+\right\rangle + \left|-\right\rangle \otimes \left|-\right\rangle) = \frac{1}{\sqrt{2}}(\left|{++}\right\rangle + \left|{--}\right\rangle) = \left|0\right\rangle_L .$$

Thus, X_L flips the logical states, acting as a logical X operator.

Commutation with stabiliser:

Both $Z_L = Z_1 Z_2$ and $X_L = X_1$ commute with the stabiliser $X_1 X_2$:

- $[Z_1 Z_2, X_1 X_2] = 0$ because Z_i and X_i anticommute on each qubit, and the anticommutations on qubits 1 and 2 cancel out (even number).
- $[X_1, X_1 X_2] = X_1(X_1 X_2) - (X_1 X_2)X_1 = X_1^2 X_2 - X_1 X_2 X_1 = X_2 - X_2 = 0.$

This confirms that Z_L and X_L are valid logical operators for the code.

Anti-commutation with each other.

The two logical operators Z_L and X_L anti-commute with one another. We can verify this for both choices of the X_L logical operator:

$$\{Z_1 Z_2, X_1\} = 0$$
$$\{Z_1 Z_2, X_2\} = 0$$

2g. What is the distance of this code?

Solution 2g:

The **distance** of a quantum code is defined as the minimum weight (number of qubits acted upon) of a non-trivial logical operator or, equivalently, the minimum weight of an undetectable error that maps codewords to other codewords without being detected by the stabilisers.

In this code:

- The stabiliser generator is X_1X_2, which stabilizes the code space spanned by $|0\rangle_L$ and $|1\rangle_L$.
- The logical Z operator is $Z_L = Z_1Z_2$, acting on both qubits.
- The logical X operator can be chosen as $X_L = X_1$ or $X_L = X_2$, each acting on a single qubit.

Minimum Weight of Logical Operators:

- $Z_L = Z_1Z_2$ has a weight of 2 since it acts non-trivially on both qubits. The Z-distance d_Z is therefore 2.
- $X_L = X_1$ or $X_L = X_2$ has a weight of 1 since it acts non-trivially on only one qubit. The X-distance d_X is therefore 1.

The minimum weight of a non-trivial logical operator is therefore 1, corresponding to the logical X operator $X_L = X_1$ or $X_L = X_2$.

Implications for Code Distance:

Since the minimum weight of a logical operator is 1, the **distance of the code is** $d = 1$.

Error Detection Capability:

- Single-qubit Z errors (phase flips) anticommute with the stabiliser X_1X_2 and are detectable.
- Single-qubit X errors (bit flips), such as X_1 or X_2, act as logical operators X_L and are undetectable by the stabiliser.

Conclusion:

This code is specifically designed to detect phase-flip errors but not bit-flip errors. The distance being 1 means that the code cannot detect all single-qubit errors, as some single-qubit errors correspond to logical operations and cannot be detected by the stabiliser.

Therefore, while the code can detect single-qubit Z errors, it cannot detect single-qubit X errors. The **distance of the code is 1**.

Problem 3: The Five-Qubit Code

The five-qubit code (also known as the 'perfect code') is defined by the stabiliser group $\mathcal{S}$ generated by $\langle S \rangle$:

$$\mathcal{S} = \langle S \rangle = \left\langle \begin{matrix} X_1 Z_2 Z_3 X_4 I_5 \\ I_1 X_2 Z_3 Z_4 X_5 \\ X_1 I_2 X_3 Z_4 Z_5 \\ Z_1 X_2 I_3 X_4 Z_5 \end{matrix} \right\rangle$$

3a. How many logical qubits are encoded by this code?

Solution 3a: The number of logical qubits k encoded by a stabiliser code is given by:

$$k = n - \text{rank}(\mathcal{S}) = n - |S|$$

For this code, $n = 5$ and $|S| = 4$. The number of logical qubits is therefore

$$k = 5 - 4 = 1$$

3b. The logical basis states of the five-qubit code are given below.

$$|0_{\text{L}}\rangle = \frac{1}{4}(|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\ -|11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle),$$

$$|1\rangle_L = \frac{1}{4}(|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle \\ -|00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle).$$

Show that both $X_L = X_1 X_2 X_3 X_4 X_5$ and $Z_L = Z_1 Z_2 Z_3 Z_4 Z_5$ are a valid choice of logical operators for the code.

Solution 3b:

We can verify this by checking the action on the logical basis states:

1. $X_L |0\rangle_L = |1\rangle_L$:

   Applying X_L flips all qubits in each term of $|0\rangle_L$, transforming it into $|1\rangle_L$.

2. $Z_L|0\rangle_L = |0\rangle_L$:

   Applying Z_L assigns a phase of $+1$ to each $|0\rangle$ and -1 to each $|1\rangle$. Due to fact each ket of $|0\rangle_L$ has an even number of '1's, the overall state remains unchanged.

3. $Z_L|1\rangle_L = -|1\rangle_L$

   Similarly, applying Z_L to $|1\rangle_L$ introduces a global phase of -1 as each ket has an odd number of '1's, consistent with the logical Z operation.

3c. Complete the single-qubit syndrome table for this code:

Error	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\mathbf{s}_4$
$X_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$Y_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$Z_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes X_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes Y_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes Z_2 \otimes I_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes X_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes Y_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes Z_3 \otimes I_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes X_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes Y_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes Z_4 \otimes I_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes X_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes Y_5$				
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes Z_5$				

Table 3: Single-Qubit Syndrome Table (Tab 3) for the Five-Qubit Code.

Solution 3c:

The completed syndrome table is shown in Tab 4.

To determine the syndrome bits $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4$ for each single-qubit error, we check the commutation relation between the error E and each stabiliser generator S_i. The syndrome bit $\mathbf{s}_i$ is set to:

$$\mathbf{s}_i = \begin{cases} 0 & \text{if } [E, S_i] = 0 \quad \text{(commute)} \\ 1 & \text{if } \{E, S_i\} = 0 \quad \text{(anticommute)} \end{cases}$$

Example:

Error $X_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$: This error anti-commutes with only S_4. This results in the syndrome $(0,0,0,1)$.

The completed table is in Tab 4.

Error	$\mathbf{s}_1$	$\mathbf{s}_2$	$\mathbf{s}_3$	$\mathbf{s}_4$
$X_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$	0	0	0	1
$Y_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$	1	0	1	1
$Z_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes I_5$	1	0	1	0
$I_1 \otimes X_2 \otimes I_3 \otimes I_4 \otimes I_5$	1	0	0	0
$I_1 \otimes Y_2 \otimes I_3 \otimes I_4 \otimes I_5$	1	1	0	1
$I_1 \otimes Z_2 \otimes I_3 \otimes I_4 \otimes I_5$	0	1	0	1
$I_1 \otimes I_2 \otimes X_3 \otimes I_4 \otimes I_5$	1	1	0	0
$I_1 \otimes I_2 \otimes Y_3 \otimes I_4 \otimes I_5$	1	1	1	0
$I_1 \otimes I_2 \otimes Z_3 \otimes I_4 \otimes I_5$	0	0	1	0
$I_1 \otimes I_2 \otimes I_3 \otimes X_4 \otimes I_5$	0	1	1	0
$I_1 \otimes I_2 \otimes I_3 \otimes Y_4 \otimes I_5$	1	1	1	1
$I_1 \otimes I_2 \otimes I_3 \otimes Z_4 \otimes I_5$	1	0	0	1
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes X_5$	0	0	1	1
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes Y_5$	0	1	1	1
$I_1 \otimes I_2 \otimes I_3 \otimes I_4 \otimes Z_5$	0	1	0	0

Table 4: Single-Qubit Syndrome Table for the Five-Qubit Code

3d. Explain why this is a correction code with distance $d \geq 3$.

Solution 3d:

From the syndrome table, we see that each single-qubit $\{X, Y, Z\}$ error maps to a unique syndrome. The number of correctable errors t is given by $t = (d-1)/2$. Rearranging this, we find that $d = 3$.

3e. Find a pair of X_L and Z_L logical operators of weight 3.

Solution 3e:

From part 3b, we have two weight-five logical operators: $X_L = X_1X_2X_3X_4X_5$ and $Z_L = Z_1Z_2Z_3Z_4Z_5$. Any logical operator multipled by a stabiliser is also a logical operator. We can therefore reduce the weight of our logicals by multiplying by stabilisers. Recall that: $XZ = -\mathrm{i}Y$.

Multiplying $S_1 = X_1Z_2Z_3X_4I_5$ by X_L gives:

$$X'_L = (X_1X_2X_3X_4X_5)(X_1Z_2Z_3X_4I_5) = -(I_1Y_2Y_3I_4X_5)$$

Similarily, multiplying $S_1 = X_1Z_2Z_3X_4I_5$ by Z_L gives:

$$Z'_L = (Z_1Z_2Z_3Z_4Z_5)(X_1Z_2Z_3X_4I_5) = -(Y_1I_2I_3Y_4Z_5)$$

The above logical operators have weight $d = 3$. The distance of the code is therefore $d = 3$.

3f. What are the $[[n, k, d]]$ parameters of this code?

Solution 3f: The number of physical qubits is $n = 5$. From part 3a, $k = 1$. From part 3e, $d = 3$. The code parameters are therefore $[[n = 5, k = 1, d = 3]]$.

Problem 4: Concatenating the $[[3, 1, 1]]$ phase-flip code into the $[[4, 2, 2]]$ code

Consider the $[[4, 2, 2]]$ code defined by the stabiliser group $\mathcal{S}$ generated by $\langle S \rangle$:

$$\mathcal{S} = \langle S \rangle = \left\langle \begin{matrix} X_1X_2X_3X_4 \\ Z_1Z_2Z_3Z_4 \end{matrix} \right\rangle,$$

and logical operator basis:

$$\mathcal{L} = \left\langle \begin{matrix} X_{L_1} = X_1I_2X_3I_4 \\ X_{L_2} = X_1X_2I_3I_4 \\ Z_{L_1} = Z_1Z_2I_3I_4 \\ Z_{L_2} = Z_1I_2Z_3I_4 \end{matrix} \right\rangle.$$

Consider also the $[[3, 1, 1]]$ phase-flip code defined by the stabiliser group $\mathcal{S}$ generated by $\langle S \rangle$:

$$\mathcal{S} = \langle S \rangle = \left\langle \begin{matrix} X_1X_2I_3 \\ I_1X_2X_3 \end{matrix} \right\rangle,$$

and logical operator basis:

$$\mathcal{L} = \left\langle \begin{matrix} X_L = X_1 I_2 I_3 \\ Z_L = Z_1 Z_2 Z_3 \end{matrix} \right\rangle.$$

4a. Write a generating set of the stabilisers of code obtained by concatenating the $[[3,1,1]]$ phase-flip code into each of the qubits of the $[[4,2,2]]$ code. E.g. replace each qubit i of the $[[4,2,2]]$ code by three qubits i_1, i_2, i_3 that form a block encoded with the $[[3,1,1]]$ phase-flip code.

Solution 4a: To concatenate the $[[3,1,1]]$ phase-flip code into each qubit of the $[[4,2,2]]$ code, we replace each qubit i of the $[[4,2,2]]$ code with three qubits i_1, i_2, i_3 that form a block encoded with the $[[3,1,1]]$ phase-flip code. To match the terminology from the lectures, the $[[4,2,2]]$ code is referred to as the outer code and the $[[3,1,1]]$ phase-flip code as the inner code. In the concatenated code, stabilisers are derived from both the inner and outer codes. The 4 blocks of the inner code each provide two stabilisers each. There are therefore 8 stabiliser that arise from the inner code, given below:

$$\begin{aligned}
S_1 &= X_1 X_2 I_3 \otimes I_4 I_5 I_6 \otimes I_7 I_8 I_9 \otimes I_{10} I_{11} I_{12} \\
S_2 &= I_1 X_2 X_3 \otimes I_4 I_5 I_6 \otimes I_7 I_8 I_9 \otimes I_{10} I_{11} I_{12} \\
S_3 &= I_1 I_2 I_3 \otimes X_4 X_5 I_6 \otimes I_7 I_8 I_9 \otimes I_{10} I_{11} I_{12} \\
S_4 &= I_1 I_2 I_3 \otimes I_4 X_5 X_6 \otimes I_7 I_8 I_9 \otimes I_{10} I_{11} I_{12} \\
S_5 &= I_1 I_2 I_3 \otimes I_4 I_5 I_6 \otimes X_7 X_8 I_9 \otimes I_{10} I_{11} I_{12} \\
S_6 &= I_1 I_2 I_3 \otimes I_4 I_5 I_6 \otimes I_7 X_8 X_9 \otimes I_{10} I_{11} I_{12} \\
S_7 &= I_1 I_2 I_3 \otimes I_4 I_5 I_6 \otimes I_7 I_8 I_9 \otimes X_{10} X_{11} I_{12} \\
S_8 &= I_1 I_2 I_3 \otimes I_4 I_5 I_6 \otimes I_7 I_8 I_9 \otimes I_{10} X_{11} X_{12}
\end{aligned}$$

The remaining stabilisers are obtained by replacing each qubit operator in the outer code stabilisers with the corresponding logical operator from the inner code. The outer code stabilisers are:

$$\begin{aligned}
S_9 &= X_1 I_2 I_3 \otimes X_4 I_5 I_6 \otimes X_7 I_8 I_9 \otimes X_{10} I_{11} I_{12} \\
S_{10} &= Z_1 Z_2 Z_3 \otimes Z_4 Z_5 Z_6 \otimes Z_7 Z_8 Z_9 \otimes Z_{10} Z_{11} Z_{12}
\end{aligned}$$

4b. How many logical qubits are encoded by this concatenated code?

Solution 4b: There are 10 independent stabilisers in total, and the concatenated code uses 12 physical qubits. The number of logical qubits k encoded by a stabiliser code is given by:

$$k = n - |\langle S \rangle| = n - |S| = 12 - 10 = 2$$

4c. Find a basis of logical operators for this concatenated code.

Solution 4c: To find a basis of logical operators for the concatenated code, we replace each qubit operator in the outer code logical operators with the corresponding logical operator from the inner code. This give four logical operators as follows:

$$\begin{aligned} X_{L_1} &= X_1 I_2 I_3 \otimes I_4 I_5 I_6 \otimes X_7 I_8 I_9 \otimes I_{10} I_{11} I_{12} \\ X_{L_2} &= X_1 I_2 I_3 \otimes X_4 I_5 I_6 \otimes I_7 I_8 I_9 \otimes I_{10} I_{11} I_{12} \\ Z_{L_1} &= Z_1 Z_2 Z_3 \otimes Z_4 Z_5 Z_6 \otimes I_7 I_8 I_9 \otimes I_{10} I_{11} I_{12} \\ Z_{L_2} &= Z_1 Z_2 Z_3 \otimes I_4 I_5 I_6 \otimes Z_7 Z_8 Z_9 \otimes I_{10} I_{11} I_{12} \end{aligned}$$

4d. What is the distance of this concatenated code?

Solution 4d: For a concatenated CSS Code, the distance is given by:

$$d = \min(d_X^{(\text{out})} d_X^{(\text{in})}, d_Z^{(\text{out})} d_Z^{(\text{in})})$$

Where $d_X^{(\text{out})}$ and $d_Z^{(\text{out})}$ are the distances of the outer code for bit-flip and phase-flip errors respectively, and $d_X^{(\text{in})}$ and $d_Z^{(\text{in})}$ are the distances of the inner code for bit-flip and phase-flip errors respectively. In this case:

$$\begin{aligned} d_X^{(\text{out})} &= 2 \\ d_Z^{(\text{out})} &= 2 \\ d_X^{(\text{in})} &= 1 \\ d_Z^{(\text{in})} &= 3 \end{aligned}$$

Therefore, the distance of the concatenated code is:

$$d = \min(2 \times 1, 2 \times 3) = \min(2, 6) = 2$$

We can also verify this by looking at the logical operators of the concatenated code found in part 4c. The minimum weight of these logical operators is 2, confirming that the distance of the concatenated code is indeed 2.

4e. What are the code parameters $[[n, k, d]]$ of this concatenated code?

Solution 4e: The code parameters $[[n, k, d]]$ of the concatenated code are:

- $n = 12$: The total number of physical qubits used in the concatenated code.
- $k = 2$: The number of logical qubits encoded by the concatenated code.
- $d = 2$: The distance of the concatenated code.

Therefore, the code parameters of the concatenated code are:

$$[[12, 2, 2]]$$

4f. The d_X distance and d_Z distance of a code are defined as the minimum weight of a logical operator consisting exclusively of Pauli X operators and Pauli Z operators respectively. What are the X-distance and Z-distance of this concatenated code?

Solution 4f: The X-distance d_X and Z-distance d_Z of the concatenated code can be determined by examining the logical operators derived in part 4c. This gives $d_X = 2$ and $d_Z = 6$.

Problem 5: The 2×2 Surface Code

5a. Figure 2 shows the Tanner graph for a surface code defined over 5 qubits. List the four stabiliser generators that are measured by this code.

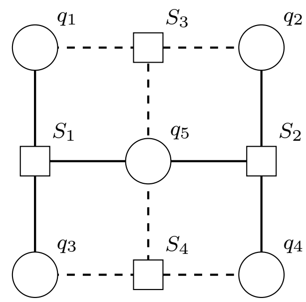

Figure 2: The five-qubit surface code. Dashed edges denote Z-type checks and solid edges X-type checks.

Solution 5a: The stabilisers of this code are:

$$S_1 = X_1X_3X_5$$
$$S_2 = X_2X_4X_5$$
$$S_3 = Z_1Z_2Z_5$$
$$S_4 = Z_3Z_4Z_5$$

5b. How many logical qubits does this code encode?

Solution 5b: There are four stabiliser generators, $|S| = 4$. The logical qubit count is therefore $k = n - |S| = 5 - 4 = 1$.

5c. This code has distance $d = 2$. Find the logical operator pair Z_L, X_L.

Solution 5c: In the surface code, logical operators span from edge-to-edge. The following is choice of logical operators:

$$X_L = X_1X_2$$
$$Z_L = Z_1Z_3$$

It is straightforward to verify that these logical operators commute with the stabilisers and anti-commute with one another. An alternative choice of logical operators is:

$$X_L = X_3X_4$$

$$Z_L = Z_2 Z_4$$

5d. Explain why this code is a detection code and not a correction code.

Solution 5d: From part 5c, we see that there are logical operators with weight 2. This code is therefore a detection code with $d = 2$. The number of correctable errors t is given by the expression $t = (d-1)/2$. As such, any correction code must have $d \geq 3$.

5e. What are the $[[n, k, d]]$ parameters of this code?

Solution 5e: The number of physical qubits $n = 5$, the logical qubit count is $k = 1$ and $d = 2$. The code has parameters $[[5, 1, 2]]$.

5f. The d_i distance of a code is defined as the minimum weight of a logical operator consisting exclusively of Pauli i operators (where $i \in \{X, Y, Z\}$). Find d_X, d_Y, and d_Z for this code.

Solution 5f: **Finding d_X and d_Z:**

From part 5c, we identified the weight-2 logical operators $X_L = X_1 X_2$ and $Z_L = Z_1 Z_3$. These are pure X-type and Z-type operators respectively, giving us $d_X = 2$ and $d_Z = 2$.

Finding d_Y:

To find d_Y, we seek a Y-only operator that (i) has zero syndrome (commutes with all stabilisers) and (ii) anti-commutes with both X_L and Z_L to act as a valid logical operator.

Consider the commutation properties of Y operators with the stabilisers:

- The stabilisers $S_1 = X_1 X_3 X_5$ and $S_2 = X_2 X_4 X_5$ are X-type
- The stabilisers $S_3 = Z_1 Z_2 Z_5$ and $S_4 = Z_3 Z_4 Z_5$ are Z-type
- Since Y anti-commutes with both X and Z, any Y operator must be carefully constructed to commute with all four stabilisers

Horizontal or vertical chains of Y operators will anti-commute with at least one stabiliser, producing a non-zero syndrome. However, diagonal chains can avoid this. For example, consider $Y_L = Y_1 Y_5 Y_4$:

- $S_1 = X_1 X_3 X_5$: intersects at qubits 1 and 5 (even) $\Rightarrow$ commutes

- $S_2 = X_2X_4X_5$: intersects at qubits 4 and 5 (even) $\Rightarrow$ commutes
- $S_3 = Z_1Z_2Z_5$: intersects at qubits 1 and 5 (even) $\Rightarrow$ commutes
- $S_4 = Z_3Z_4Z_5$: intersects at qubits 4 and 5 (even) $\Rightarrow$ commutes

We can verify that $Y_1Y_5Y_4$ anti-commutes with both $X_L = X_1X_2$ and $Z_L = Z_1Z_3$, confirming it acts as a logical operator. Similarly, $Y_2Y_5Y_3$ is another weight-3 Y-type logical operator.

Since no weight-2 Y-only operator can satisfy the required commutation relations, we conclude $d_Y = 3$.

Summary: $d_X = 2$, $d_Z = 2$, and $d_Y = 3$.

Problem 6: The 4×4 Surface Code

6a. Figure 3 shows the Tanner graph for a distance-4 surface code. Two X-errors have occurred on qubits q_{20} and q_6 activating a non-zero syndrome measurement for stabilisers S_{17} and S_{19}. Explain why $\mathcal{R} = X_6X_{20}$ and $\mathcal{R}' = X_{10}X_{21}$ are both suitable recovery operations.

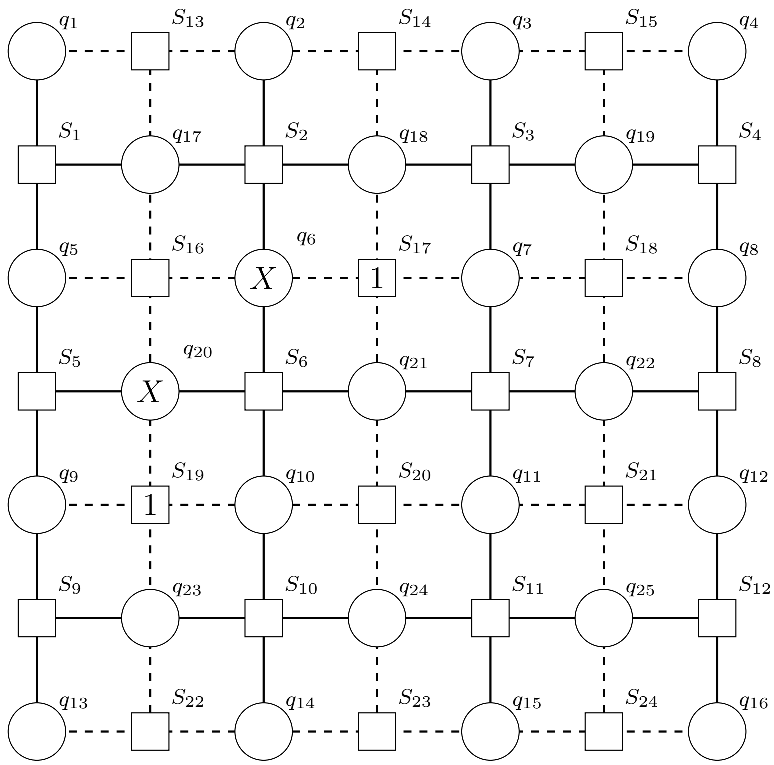

Figure 3: The distance-4 surface code. Dashed edges denote Z-type checks and solid edges X-type checks.

Solution 6a: The original error is $E = X_6X_{20}$. The determine whether or not our recovery operation is successful we first calculate the residual error. For the first recovery operation $\mathcal{R}$ is this given by:

$$R = \mathcal{R}E = I \in \mathcal{S}$$

which is in the stabiliser group. For the second recovery $\mathcal{R}'$ the residual error is:

$$R' = \mathcal{R}'E = X_6X_{10}X_{20}X_{21} \in \mathcal{S}$$

This residual is equivalent to a stabiliser as it is equal to the operator measured by generator S_6.

6b. The recovery operator $\mathcal{R}'' = X_7X_8X_9$ would also reset the total syndrome of the surface code. Explain why this is not a suitable recovery operator.

Solution 6b: The residual error for this recovery would be:

$$R'' = \mathcal{R}''E = X_6X_7X_8X_9X_{20} \in \mathcal{L}$$

This represents a chain of X-type Pauli operators spanning from the left edge of the surface code to the right edge. Error chains of this type are equivalent to X_L logical operators. The recovery operator $\mathcal{R}''$ would therefore change the logical information encoded by the code.

Problem 7: The Rotated Surface Code

The Tanner graph for a 5×5 rotated surface code is shown in Figure 4.

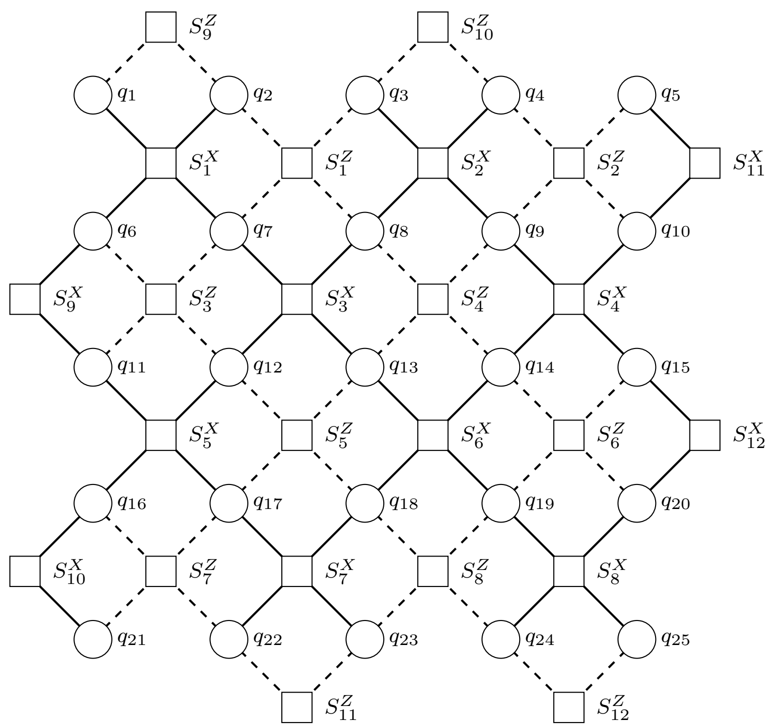

Figure 4: The Tanner graph for a 5×5 rotated surface code. Dashed lines represent Z-type Pauli operators and solid X-type Pauli operators.

7a. How many logical qubits does this code encode?

Solution 7a: There are 25 data qubits, so $n = 25$. There are 12 X-type stabilisers and 12 Z-type stabilisers, giving a total of $|S| = 24$ stabilisers. The number of logical qubits is therefore $k = n - |\langle S \rangle| = 25 - 24 = 1$.

7b. Find weight-5 logical operators X_L and Z_L for this code.

Solution 7b: In the rotated surface code, logical operators span from edge-to-edge. A suitable choice of weight-5 logical operators is:

$$X_L = X_1 X_2 X_3 X_4 X_5$$

$$Z_L = Z_1 Z_6 Z_{11} Z_{16} Z_{21}$$

7c. The *footprint* of a QEC code is defined as the total qubit count: data qubits plus auxiliary/measurement qubits. What is the ratio of footprints for the distance-d rotated surface code to the distance-d standard surface code?

Solution 7c: The *standard* surface code of distance d uses $d^2 + (d-1)^2$ data qubits and $d^2 + (d-1)^2 - 1$ measurement qubits, giving a total footprint of:

$$\mathcal{F}_S = 2(d^2 + (d-1)^2) - 1 = (2d-1)^2$$

Similarly, the *rotated* surface code of distance d uses d^2 data qubits and $(d)^2 - 1$ measurement qubits, giving a total footprint of:

$$\mathcal{F}_R = d^2 + d^2 - 1 = 2d^2 - 1$$

The ratio of footprints is therefore:

$$\mathcal{R}_{\frac{\mathcal{F}_R}{\mathcal{F}_S}} = \frac{\mathcal{F}_R}{\mathcal{F}_S} = \frac{2d^2 - 1}{(2d-1)^2}$$

Problem 8: Implementing the 7-qubit Steane Code in Pennylane

The 7-qubit Steane code is defined by the stabiliser group $\mathcal{S}$ generated by:

$$\mathcal{S} = \langle S \rangle = \left\langle \begin{matrix} I_1 I_2 I_3 X_4 X_5 X_6 X_7 \\ I_1 X_2 X_3 I_4 I_5 X_6 X_7 \\ X_1 I_2 X_3 I_4 X_5 I_6 X_7 \\ I_1 I_2 I_3 Z_4 Z_5 Z_6 Z_7 \\ I_1 Z_2 Z_3 I_4 I_5 Z_6 Z_7 \\ Z_1 I_2 Z_3 I_4 Z_5 I_6 Z_7 \end{matrix} \right\rangle,$$

8a. Show that the $|0\rangle_L$ logical state of the Steane code can be prepared using the following operation on the blank state:

$$|0\rangle_L = \prod_{P \in \langle S \rangle} \frac{I + P_i}{\sqrt{2^{|\langle S \rangle|}}} |0000000\rangle$$

Solution 8a: First, we know that for any pair i, j the operators $(1 + P_i)$ and $(1 + P_j)$ commute. This is because all stabilisers commute with each other. Now, for any stabiliser P_i we can move the correspoding $(I + P_i)$ operator to the front of the product:

$$|0\rangle_L = \frac{I + P_i}{\sqrt{2}} \prod_{P \in S, P \neq P_i} \frac{I + P_j}{\sqrt{2^{|S|-1}}} |0000000\rangle$$

To show that any stabiliser P_i stabilises the logical state $|0\rangle_L$, we can apply P_i to the above expression:

$$P_i |0\rangle_L = P_i \frac{I + P_i}{\sqrt{2}} \prod_{P \in S, P \neq P_i} \frac{I + P_j}{\sqrt{2^{|S|-1}}} |0000000\rangle = \frac{P_i + P_i^2}{\sqrt{2}} \prod_{P \in S, P \neq P_i} \frac{I + P_j}{\sqrt{2^{|S|-1}}} |0000000\rangle$$

Since $P_i^2 = I$, we have:

$$P_i |0\rangle_L = \frac{I + P_i}{\sqrt{2}} \prod_{P \in S, P \neq P_i} \frac{I + P_j}{\sqrt{2^{|S|-1}}} |0000000\rangle = |0\rangle_L$$

I.e., the state remains unchanged when acted on by any stabiliser P_i. Therefore, $|0\rangle_L$ is stabilised by all elements of the stabiliser group $\mathcal{S}$.

8b. The circuit shown in Figure 5 can be used to map the blank state $|0000000\rangle$ onto the $+1$ eigenspace of the X-type stabiliser $IIIXXXX$.

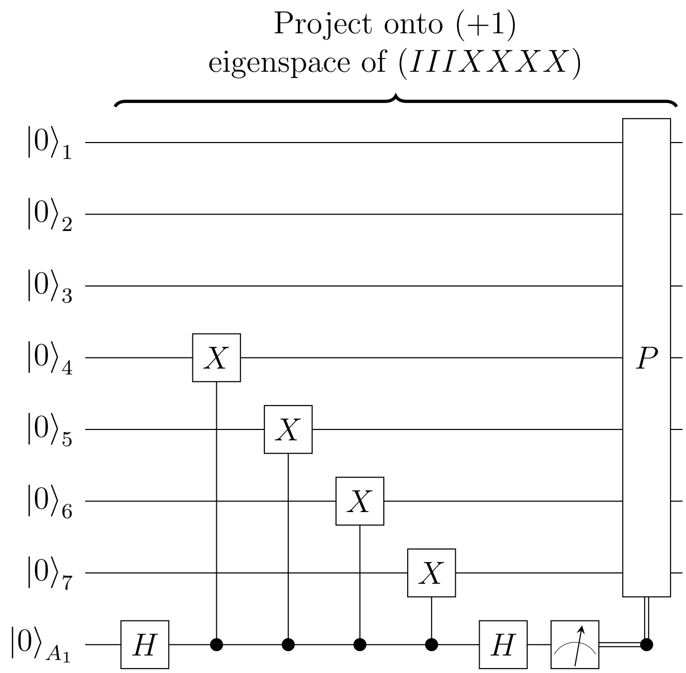

Figure 5: Circuit to map the blank state $|0000000\rangle$ onto the $+1$ eigenspace of the X-type stabiliser $IIIXXXX$.

Explain why the classical feedback is required to ensure the output state is in the $+1$ eigenspace. Find an appropiate form of the feedback operation P.

Solution 8b: Just before measurement of the auxiliary qubit A_1 the system is in the state:

$$\frac{1}{2}(1 + IIIXXXX)\,|0000000\rangle\,|0\rangle_{A_1} + \frac{1}{2}(1 - IIIXXXX)\,|0000000\rangle\,|1\rangle_{A_1}$$

As such, when the auxiliary qubit is measured, there is a 50% chance of obtaining the measurement result 0 and projecting the data qubits onto the $+1$ eigenspace of the stabiliser $IIIXXXX$. There is also a 50% chance of obtaining the measurement result 1 and projecting the data qubits onto the -1 eigenspace of the stabiliser $IIIXXXX$. To ensure that the output state is always in the $+1$ eigenspace, we need to apply a correction operation P when the measurement result is 1. This correction operation must anti-commute with the stabiliser $IIIXXXX$ to flip the eigenvalue from -1 to +1. An appropriate choice for P is therefore $P = IIIIIIZ$.

8c. Implement the full encoding circuit for the 7-qubit Steane code in Pennylane. You may use the circuit from Figure 5 as a subroutine and repeat for all stabilisers. *Hint:* You can use the function `qml.cond` to implement classical feedback based on measurement results. To verify that the stabiliser readout is deterministic, implement a full round of stabiliser measurements after the encoding circuit. Then check that all stabiliser measurements are deterministic by measuring out the auxiliary qubits with the `qml.sample` function for multiple shots.

8d. By inserting Pauli-errors after the encoding circuit, verify that your implementation can detect all single-qubit errors by creating the corresponding syndrome table.

Solution 8d:

An example script can be downloaded from `https://gist.github.com/quantumgizmos/20910d5c46c9a3c9028298e61c229e16`